1) $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Minor of order 3

$|A| = -5 \neq 0$

$\rho(A)$ = Rank of $A = 3$

2) $A = \begin{bmatrix} \lambda & 1 & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix}$

$\rho(A) < 3 \Rightarrow |A| = 0$

$$\begin{vmatrix} \lambda & 1 & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda^2 - 1) - \lambda = 0$$

$$\lambda^3 - 2\lambda = 0$$

$$\lambda(\lambda^2 - 2) = 0$$

$$\lambda = 0, \pm\sqrt{2}$$

3) $A = \begin{bmatrix} 2 & -1 & 3 \\ 8 & 2 & 0 \\ 0 & 1 & k \end{bmatrix}$

$\rho(A) < 3 \Rightarrow |A| = 0$

$$\begin{vmatrix} 2 & -1 & 3 \\ 8 & 2 & 0 \\ 0 & 1 & k \end{vmatrix} = 0 \Rightarrow \boxed{k = -2}$$

for $k = -2$, $A = \begin{bmatrix} 2 & -1 & 3 \\ 8 & 2 & 0 \\ 0 & 1 & -2 \end{bmatrix}$

Minor of order 2, $\begin{vmatrix} 2 & -1 \\ 8 & 2 \end{vmatrix} \neq 0$

$\Rightarrow \rho(A) = 2$.

4) $$A = \begin{bmatrix} x+a & a & a \\ b & x+b & b \\ c & c & x+c \end{bmatrix}$$

$\rho(A) < 3 \Rightarrow |A| = 0$

$$\begin{vmatrix} x+a & a & a \\ b & x+b & b \\ c & c & x+c \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$(x+a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & x+b & b \\ c & c & x+c \end{vmatrix} = 0$$

$R_2 \to R_2 - bR_1, \quad R_3 \to R_3 - cR_1$

$$(x+a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = 0$$

$$(x+a+b+c)x^2 = 0$$

$$x = 0, 0, -(a+b+c)$$

OR

5) Find value of $x \Rightarrow \rho(A) < 3$

$$\begin{bmatrix} x+a & a & a \\ b & x+b & b \\ c & c & x+c \end{bmatrix}$$

let, $S = a+b+c$

$$\det(A) = (x+a)\left[(x+b)(x+c) - (bc)\right] - a\left[(x+c)(b) - (bc)\right] + a\left((bc) - c(x+b)\right) = 0$$

$$= x^2(x+S) = 0$$

$\therefore x = 0$ or $-(a+b+c)$

If $a = b = c \Rightarrow$ rank $\rho(A) = 0$

If rows are proportional $\rho(A) = 1$

If $x = -(a+b+c)$ $\rho(A) = 2$

6) $\rho(A) = ?$ using echelon form.

$$A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 4 & -2 & 1 \\ 2 & -1 & 3 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & -9 & 7 & -1 \\ 0 & -18 & 14 & -2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & -9 & 7 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ No. of non zero rows $= 2$

$\therefore \rho(A) = 2$

Q 7) $f(A) = ?$

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

(i) $R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

(ii) $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

(iii) $R_3 \to R_3 - \frac{4}{5} R_2$

$R_4 \to R_4 - \frac{9}{5} R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & \frac{33}{5} & 22/5 \end{bmatrix}$$

(iv) $R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore f(A) = 3$

8) Reduce matrix to normal form & hence find their rank

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \frac{1}{2} R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$\therefore \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$

9) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ Normal form & $\rho(A)$?

$C_2 \leftrightarrow C_1$

$$\begin{bmatrix} 1 & 8 & 3 & 6 \\ 3 & 0 & 2 & 2 \\ -1 & -8 & -3 & 4 \end{bmatrix}$$

$R_3 \to R_3 + R_1$; $R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 8 & 3 & 6 \\ 0 & -24 & -7 & -16 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$R_3 \to \frac{1}{10} R_3$

$$\begin{bmatrix} 1 & 8 & 3 & 6 \\ 0 & -24 & -7 & -16 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 6R_3$; $R_2 \to R_2 + 16R_1$

$$\begin{bmatrix} 1 & 8 & 3 & 0 \\ 0 & -24 & -7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 + \frac{1}{3} R_2$

$$\begin{bmatrix} 1 & 0 & 2/3 & 0 \\ 0 & -24 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + \frac{7}{24} C_2$

$$\begin{bmatrix} 1 & 0 & 2/3 & 0 \\ 0 & -24 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - \frac{2}{3} C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -24 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to \frac{-1}{24} C_2$

$C_3 \leftrightarrow C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore [I_3 \; 0]$

$\therefore \rho(A) = 3$

***

10) Reduce to normal form & find rank

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$

$$\begin{bmatrix} 1 & 0 & -1 & 4 \\ 2 & 0 & 3 & 4 \\ 1 & 0 & 3 & 4 \\ -1 & 0 & 6 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$

$R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

$C_2 \leftrightarrow C_3$

$$\begin{bmatrix} 1 & -1 & 0 & 4 \\ 0 & 5 & 0 & -4 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix}$$

$R_1 \to R_1 + \frac{1}{5} R_2$

$R_3 \to R_3 - \frac{4}{5} R_2$

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 16/5 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 0 & 16/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - \frac{16}{5} R_4$

$R_2 \to R_2 + 4R_4$

$R_3 \to R_3 - \frac{16}{5} R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \leftrightarrow C_3$

$C_2 \to \frac{1}{5} C_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \therefore \rho(A) = 3.$$